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Unsteady Boundary Layer Rotating Flow and Heat Transfer in a Copper-water Nanofluid over a Stretching Sheet

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ABSTRACT

This paper discussed the study of unsteady three-dimensional boundary layer rotating flow and heat transfer in Copper-water nanofluid over a stretching sheet. The partial differential equations are transformed to ordinary differential equations using the appropriate similarity variables and are then solved numerically by shooting method. The effects of unsteady parameter A, suction and injection parameter s, rotating parameter Ω and volume fraction parameter φ on the skin friction coefficient, local Nusselt number together with dual velocity as well as temperature profiles are shown graphically

Keywords: Rotating flow, stretching sheet and nanofluid.

1. Introduction

Boundary layer theory was proposed by Ludwig Prandtl in 1904 Tulapurkara (2005) This theory gave an initial probation on solving hydrodynamic studies and also being widely applied in engineering, industry and earth system for a better optimization. Due to these applications, the research on boundary layer flow due to stretching and shrinking sheet problems became an interesting area to study on.

Recently, the unsteady boundary layer problems have been studied widely which the parameters in the problems are time-dependent where the problems have been studied in various surfaces, effects, fluids, boundary layer conditions and so on. Some researchers have considered the stagnation-point flow in their works for instance by Ali et al. (2011), Fan et al. (2010), Fang et al. (2011), Labropulu (2011) , Mahapatra and Nandy (2011), Malvandi (2015) and Suali et al. (2012). Besides, the presence of different effects in the unsteady boundary layer flow has been investigated in a few papers for instance Ali et al. (2014) discussed the presence of the radiation effect in their study while the effect of thermal radiation is studied by Das et al. (2014) and Labropulu (2011) has considered the effect of a magnetic field. As for stretching or shrinking sheet problems,Surma Devi et al. (1986) initiated the study on unsteady boundary layer flow over a stretching sheet and followed by Pop and Na (1996). Ishak et al. (2009) considered the mixed convection and heat transfer over vertical surface while Fang et al. (2010) introduced a new class of boundary layer flow over a constant speed stretching surface from a slot.

A part from that, the studies on boundary layer flow induced by shrinking sheet have been done by Sajid et al. (2008) by considering magneto-hydrodynamic (MHD) rotating flow. Then Bachok et al. (2010) studied the three-dimensional flow and heat transfer due to shrinking sheet in two lateral directions. Rohni et al. (2013) investigated the shrinking problem with mass suction in a nanofluid for different nanoparticles using a single-phase model. Other than nanofluid, Yacob et al. (2012) applied non-Newtonian power-law fluid in their work.

However, there are only a few studies about the unsteady boundary layer flow induced by stretching sheet in rotating fluid so far. To date, this study is initially discussed by Rajeswari and Nath (1992) where the effect of power-law variation on heat transfer characteristics is considered. Nazar et al. (2004) studied the unsteady flow when the rotating fluid is caused by the suddenly stretched surface. Further, Abbas et al. (2010) have considered MHD boundary layer and heat transfer on the stretching sheet in rotating fluid.

The purpose of this study is to extend the work of Ali et al. (2011) to the case of stretching sheet in the Copper-water nanofluid. The partial differential equations are transformed to ordinary differential equations by introducing the similarity variables. The problem is solved numerically using a shooting method to obtain the numerical solutions.

2. Basic Equation

Consider the unsteady laminar flow over a permeable stretching surface in a rotating nanofluid. The nanofluid motion is three-dimensional since the Cariolis force exists in the present problem. The Cartesian coordinates are x,y and z where the fluid is rotating at an angular velocity about the z - axis where time is denoted as t .

Let the velocities in the (x,y) directions are $u_w(x,t)$ and $v_w(x,t)$, respectively and the wall mass flux velocity in the z -direction is $w_w(x,t)$. Under these conditions, the Navier-Stokes equations in the Cartesian coordinates system are given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-2\overline{\Omega}v=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 u}{\partial z^2} \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\overline{\Omega}u=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 v}{\partial z^2} \tag{3}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^2 w}{\partial z^2} \tag{4}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\alpha_{nf}\frac{\partial^2 T}{\partial z^2} \tag{5}$$

subject to the initial and boundary conditions

$$\begin{gathered} t<0: u=v=w \text{ for all } x,y,z \\ t\geq 0: u=u_w(x,t),\ v=v_w(x,t),\ w=w_w(x,t),\ T=T_w \text{ at } z=0 \\ u\to 0,\quad v\to 0\quad as\quad z\to\infty \end{gathered} \tag{6}$$

where u, v and w are the velocity components in x, y and z directions, $\overline{\Omega}$ is the constant angular velocity of the nanofluid, μ_{nf} is the dynamic viscosity of the nanofluid, $\rho_{nf}$is the density of the nanofluid, α_{nf} is the thermal diffusivity of the nanofluid, T is the temperature of nanofluid, T_w is the wall temperature, $w = w_w(x,t) < 0$ for suction and $w = w_w(x,t) > 0$ for injection . All parameters μ_{nf} , ρ_{nf} and α_{nf} are related with nanoparticle volume fraction, φ that can be defined as

$$\rho_{nf} = \rho_f\left[1-\varphi+\varphi\left(\frac{\rho_s}{\rho_f}\right)\right], \quad \mu_{nf} = \frac{\mu_f}{(1-\varphi)^{2.5}},$$
$$\alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}}, \quad (\rho c_p)_{nf} = (\rho c_p)_f\left[1-\varphi+\varphi\left(\frac{(\rho c_p)_s}{(\rho c_p)_f}\right)\right], \tag{7}$$
$$\frac{k_{nf}}{k_f} = \frac{k_s+2k_f-2\varphi(k_f-k_s)}{k_s+2k_f+\varphi(k_f-k_s)}$$

where $(\rho c_p)_s$ is the volumetric heat capacity of the solid nanoparticles, $(\rho c_p)_p$ is the volumetric heat capacity of the base fluid, $(\rho c_p)_{nf}$ is the volumetric heat capacity of the nanofluid, k_{nf} is the thermal conductivity of the nanofluid, k_f is the thermal conductivity of the base fluid, k_s is the thermal conductivity of the solid nanoparticles, φ is the nanoparticle volume fraction, ρ_f is the density viscosity of the base fluid and μ_f is the dynamic viscosity of the base fluid.

In order to reduce the partial differential equations (1) - (5) to ordinary differential equations, we assume that $u_w(x,t)$, $v_w(x,t)$ and $\overline{\Omega}(t)$ have the following forms:

$$u_w(x,t) = \frac{ax}{1-\delta t}, \; v_w(x,t) = -\frac{ax}{1-\delta t}, \; \overline{\Omega}(t) = \frac{\omega}{1-\delta t} \tag{8}$$

where a is the stretching rate, ω is the constant angular velocity of the stretching sheet and δ represents the unsteadiness parameter.

We introduce the following similarity variables:

$$u = \frac{ax}{1-\alpha t}f'(\eta), \; v = \frac{ax}{1-\alpha t}g(\eta), \; w = -\sqrt{\frac{av}{1-\alpha t}}f(\eta),$$
$$\eta = \sqrt{\frac{a}{v(1-\alpha t)}}z, \; \theta(\eta) = \frac{T-T_\infty}{T_w-T_\infty} \tag{9}$$

Thus, form of $w_w(x,t)$ is defined as

$$w_w(x,t) = -s\sqrt{\frac{av}{1-\delta t}} \tag{10}$$

where s is the constant wall transfer parameter with for $s > 0$ suction and $s < 0$ for injection.

Using similarity variables (9), Equation (1) is identically satisfied. Substituting Equations (9) and (10) into (2), (3) and (5), we obtain the following ordinary differential equations:

$$\frac{1}{(1-\varphi)^{2.5}\left[1-\varphi+\varphi\left(\frac{\rho_s}{\rho_f}\right)\right]}f''' - \left[f'^2 - ff'' - 2\Omega g + A\left(\frac{\eta}{2}f'' + f'\right)\right] = 0 \tag{11}$$

$$\frac{1}{(1-\varphi)^{2.5}\left[1-\varphi+\varphi\left(\frac{\rho_s}{\rho_f}\right)\right]}g'' - \left[f'g - fg' + 2\Omega f' + A\left(\frac{\eta}{2}g' + g\right)\right] = 0 \tag{12}$$

$$\frac{k_{nf}}{k_f\left[1-\varphi+\varphi\left(\frac{(\rho c_p)_s}{(\rho c_p)_f}\right)\right]}\theta'' - Pr\left[A\frac{\eta}{2}\theta' - f\theta'\right] = 0 \tag{13}$$

subject to the boundary conditions

$$\begin{gathered} f(0) = s, \quad f'(0) = 1, \quad g(0) = -1, \quad \theta(0) = 1 \quad \text{at} \quad \eta = 0 \\ f'(\eta) \to 0, \quad g(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \end{gathered} \tag{14}$$

where $\Omega = \omega/a$ is an important non-dimensional parameter signifying the relative importance of rotation rate to stretching rate and $A = \delta/a$ is unsteadiness parameter. Primes denote the differentiation with respect to η. Roşca and Pop (2013) mentioned that the pressure term (p) can be integrated from (4) and we get

$$p = \nu\rho\frac{\partial w}{\partial z} - \rho\frac{w^2}{2} + c \tag{15}$$

The physical quantities of interest in this problem are the skin friction coefficients in the x and y directions, C_{fx} and also local Nusselt number, Nu_x that can be defined as

$$C_{fx} = \frac{\tau_{wx}}{\rho u_w^2(x,t)} \quad \text{and} \quad Nu = \frac{xq_w}{k_f(T_w - T_\infty)} \tag{16}$$

where

$$\tau_{wx} = \mu_{nf}\left(\frac{\partial u}{\partial z}\right)_{z=0} \quad \text{and} \quad q_w = -k_{nf}\left(\frac{\partial T}{\partial z}\right)_{z=0} \tag{17}$$

Using Equations (16) and (17), we obtain

$$\sqrt{Re_x}C_{fx} = \frac{1}{(1-\varphi)^{2.5}}f"(0) \quad \text{and} \quad \frac{Nu}{\sqrt{Re_x}C_{fx}} = \frac{k_{nf}}{k_f}[-\theta'(0)] \tag{18}$$

where $Re_x{=}u_w x/v$ is local Reynold number.

3. Results and Discussion

The system of ordinary differential equations (11) - (13) subject to boundary conditions (14) is solved numerically using a shooting method in Maple software for some values of rotating parameter Ω and nanoparticle volume fraction, φ. Prandtl number is fixed to 6.2 (water) and values of φ are between 0 and 0.2 (Oztop and Abu-Nada (2008)). The thermophysical properties of the fluid and the nanoparticle, Copper, as shown in Table 1.

Table 1: Thermophysical properties of fluid and selecred nanoparticle (Oztop and Abu-Nada (2008)).

Physical properties	Fluid phase(water)	Cu
C_p(J/kg K)	4179	385
ρ(kg/m)	997.1	8933
k(W/mk)	0.613	400

The solutions are obtained for the missing values of $f''(0)$ and -θ'(0) where all profiles satisfy the boundary conditions (14) asymptotically by guessing the initial values. This study analysed the effects of the rotating parameter and nanoparticle volume fraction on the skin friction, local Nusselt number, and dual velocity and temperature profiles for Copper-water nanofluid.

Apparently dual solutions do exist in stretching sheet as presented in Figures 1 - 7. The effect of nanoparticle volume fraction, φ and rotating parameter, Ω with respect to mass suction/ injection, s and unsteadiness parameter, A are investigated as shown in Figures 1 - 4.

Figures 1(a), 2(a), 3(a) and 4(a) represent the variation of skin friction coefficients $f''(0)$ with s and A for various values of φ and Ω, respectively. These figures show that the values of $f''(0)$ are always negative indicates that the wall drags the Copper-water nanofluid. Besides, we find that the increase in nanoparticle volume fraction and rotation parameters decrease the skin friction coefficient, respectively.

The variations of local Nusselt number -θ'(0) towards s and A are displayed in Figures 1(b), 2(b), 3(b) and 4(b). As shown in Figures 1(b) and 3(b), both of the variations represent the effects of φ on s and A, respectively where the local Nusselt number increases when nanoparticle volume fraction is increased. However, the effects of on s and A show the opposite as shown in Figures 2(b) and 4(b). Figure 2 also showed that the dual solutions do exist in stretching sheet problem in certain range of rotation parameter, $0 \leqslant \Omega < 0.5$. It is clear from Figure 2 when Ω=0.5 , only one solution we get.

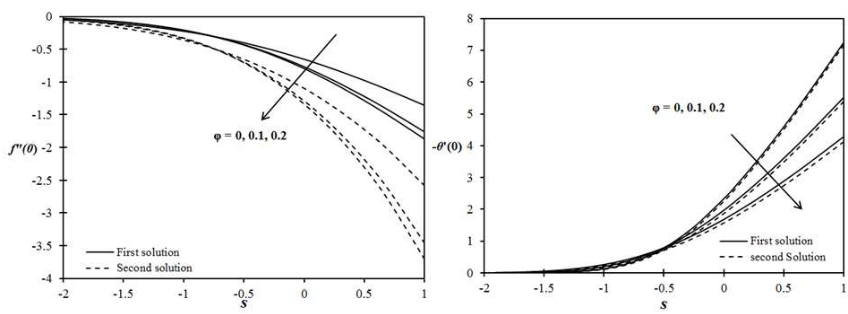

Figure 1: (a) - Variation of $f''(0)$ with different s for various φ when Ω = 0.001 and A = -1 ; (b) - Variation of - θ '(0) with different s for various φ when Ω = 0.001 and A = -1.

Figures 5 (a) and (b) illustrate the variation of skin friction coefficient and local Nusselt number given by Equation (18) with the different values of rotating parameter Ω for $0 \leqslant \varphi \leqslant 0.2$. Both figures show that the skin friction coefficient and the local Nusselt number decrease with increasing φ as can be seen in Figure 5. Also, both of that decrease with increasing Ω.

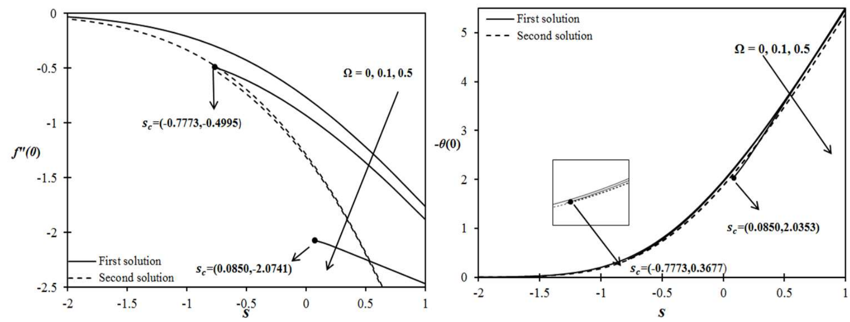

Figure 2: (a) - Variation of $f''(0)$ with different s for various Ω when $\varphi = 0.1$ and $A = -1$; (b) - Variation of - θ '(0) with different s for various Ω when $\varphi = 0.1$ and $A = -1$.

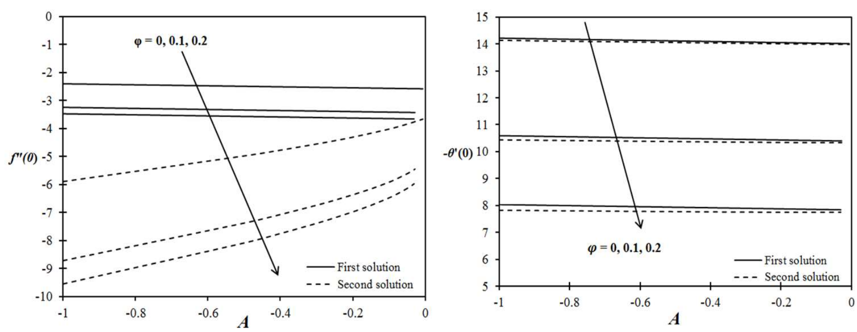

Figure 3: (a) - Variation of $f''(0)$ with different A for various φ when $\Omega = 0.001$ and $s = 2.2$; (b) - Variation of - θ '(0) with different A for various φ when $\Omega = 0.001$ and $s = 2.2$.

The dual solutions as presented in Figures 1 - 5 is supported by Figures 6 - 7 that show the dual velocity and temperature profiles. According to these figures, the far field boundary conditions by Equation (14) is satisfied for first and second solutions. Hence, the obtained numerical results are supported. Figure 6 shows that the velocity boundary layer thickness for second solutions is smaller than first solutions but the boundary layer thickness are different for temperature profiles where first solutions are smaller than second solution in all cases as illustrated in Figure 7. A part from that, the increasing of nanoparticle volume fraction, rotating and unsteadiness parameter decreases the velocity boundary layer thickness for both solutions in Figure 6. Figure 7 illustrates the effect of φ, Ω and A on temperature profile. The results show that the increasing of φ and Ω parameter increase the temperature boundary

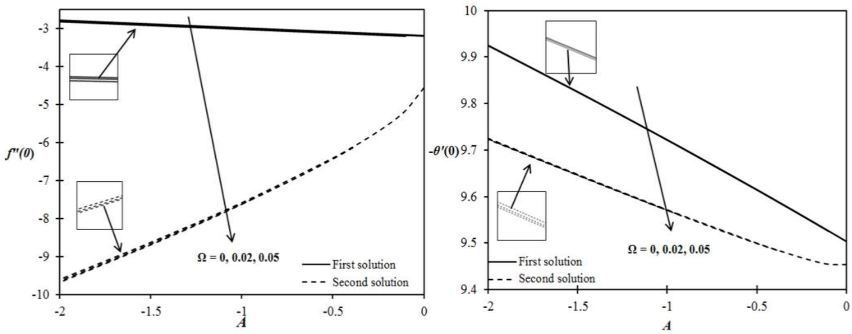

Figure 4: (a) - Variation of $f''(0)$ with different A for various Ω when $\varphi = 0.1$ and $s = 2$; (b) - Variation of - θ '(0) with different A for various Ω when $\varphi = 0.1$ and $s = 2$.

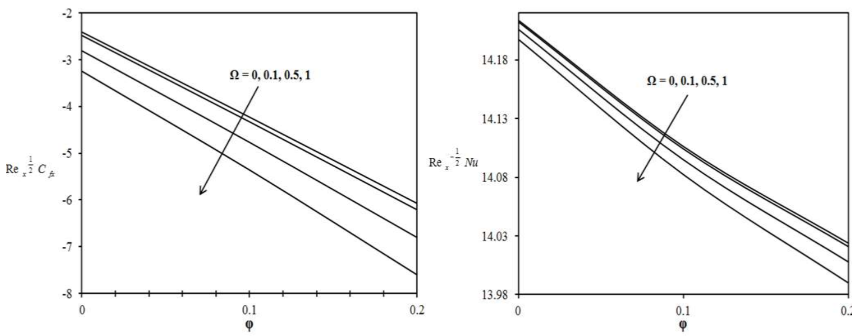

Figure 5: (a) - Variation of $Re_x^{1/2}C_f$ with φ for various values of Ω ; (b) - Variation of - Variation of $Re_x^{-1/2}Nu_x$ with φ for various values of Ω.

layer thickness but the profile decreases when the parameter A increases. Since we have dual solutions which is first solution and second solution in this present work, the solution that is stable and physically reliable as mentioned by some researches in their research such as Nazar et al. (2014) and Hafidzuddin et al. (2015) is first solution while second solution is not stable.

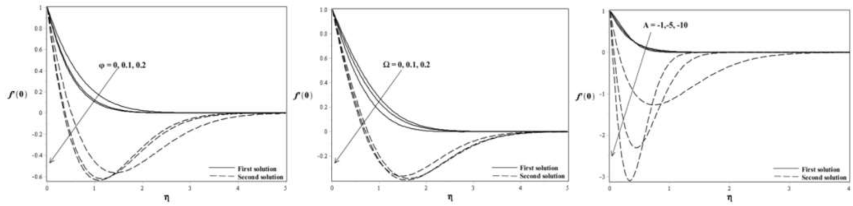

Figure 6: (a) - Dual velocity profiles $f'(\eta)$ for various values of φ; (b) - Dual velocity profiles $f'(\eta)$ for various values of Ω; (c) - Dual velocity profiles $f'(\eta)$ for various values of A.

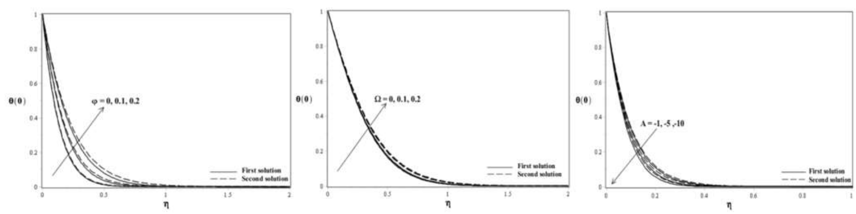

Figure 7: (a) - Dual velocity profiles $\theta(\eta)$ for various values of φ; (b) - Dual velocity profiles $\theta(\eta)$ for various values of Ω; (c) - Dual velocity profiles $\theta(\eta)$ for various values of A.

4. Concluding Remarks

The problem of unsteady boundary layer flow over stretching sheet in rotating Copper-water nanofluid was numerically studied. The effects of several considered parameters on the velocity and temperature profiles as well as the skin friction coefficient and local Nusselt number were discussed and shown graphically. This study showed that the dual solutions do exist in stretching sheet problem in certain range of rotation parameter, $0 \leqslant \Omega < 0.5$. The skin friction coefficients were seen to be decreased as the mass suction/ injection, unsteadiness parameters, nanoparticle volume fraction and rotating parameter increase. Further, the local Nusselt number was found to be decreased as rotating parameter and nanoparticle volume fraction increased.

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